The effect of clusterings on the equilibrium states of local majority-rule: Occurrence probability and robustness

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1 Introduction

The collective behaviors, such as the emergence of system-wide coordination in nature, the appearance of the consensuses of opinions in social systems, and other related phenomena, are of great interested for many researchers[1, 2, 3, 4, 5]. The local majority-rule (LMR) has often been employed to study the arising of such behaviors. The rule is simple, based on the principle of majority vote without much consideration in psychological level; it dictates that the time evolution of the state of an unit (individual or agent) is determined by the majority-favored state of its neighbors[6]. The neighbors of an unit can be given by geographic, cultural, social, or organizational proxim-

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ity, here we use artificially constructed networks to define the neighbors of an unit as its nearestly connected nodes. Because of the locality in LMR, one may expect that the distribution of cliques of a system can affect the occurrence probability of collective behavior, and this work is devoted to analyze such effect.

The global topology of a network can be characterized by two quantities, the degree distribution and the clustering coefficient[7]. The total number of connections of a node is referred as the degree of the node k, and the probability that a randomly chosen node has k connections is given by the degree distribution $P(k)$; the tightness of a clique formed by a site and its directly connected neighbors can be characterized globally by the clustering coef-

ficient of a network C. The question concerning with the role of network topology on dynamical cooperative behavior were discussed by Sood and Render[8] and by Suchecki et al.[9] in the voter model which may be viewed as a statistical model of LMR[10]. The mean time for reaching the state of collective behavior was shown to have different scaling behaviors with respect to the number of nodes for different decay exponents γ of scale-free (SF) networks between $\gamma > 3$, $\gamma = 3$, $2 < \gamma < 3$, $\gamma = 2$, and $\gamma < 2$[8]. Moreover, network geometry was also shown to have important effect on the dynamics such as the average survival time of metastable states in finite networks, the linear size scaling law of the survival time, and the size of an ordered domain[9]. As LMR is the root of the voter model, these features may be traced to the properties of the equilibrium states of LMR. Hence the study within LMR may provide more insights to the question.

Different γ values of SF networks characterize the difference in the appearance of hub-nodes. Here the hub-nodes are referred to those possessing large degree of connections. The existence of hub-nodes may affect strongly the efficiency of reaching an equilibrium state of LMR. For this aspect, Zhou and Lipowsky showed that there exists categorical difference between the SF networks of $\gamma < 5/2$ and those of $\gamma > 5/2$ for the scaling behavior of the relaxation time from a strongly disorder state towards an order state[13]. But there are other equilibrium states associated with LMR that are different characteristically from the state of collective behavior. Moreover, as a system evolves from an initial state, the distribution

of cliques in the system affects the corresponding trajectory strongly, and hence affects the type of equilibrium states reached by the trajectory. One of the questions we intend to address in this work is as follows: Starting with strongly disorder states, what role does the clustering coefficient of the system play in the type of the equilibrium states reached by the system?

Another question that attracts our attention is the relation between the robustness of a equilibrium state and the clustering coefficient of a system. The study in this aspect may not only reveal the stability of an equilibrium state but also provide an estimation for the external strength required to break the state. The robustness of an equilibrium state can be characterized by its escape rate after introducing fluctuation to perturb the system. In fact, fluctuation is an unavoidable component for real systems. The attempt was made by Moreira et al. to include noise into the dynamics by changing the transition probability of LMR from 1 to $1 - \eta$, where the parameter η characterizes the average effect of fluctuation[14]. The authors showed that the presence of fluctuation may increase the probability and the efficiency of occurring collective behavior for systems with small-world characters. In this work, we take a microscopic approach by proposing a stochastic LMR in which, each node-state contains a component of white noise. As the equilibrium states of LMR become transient in stochastic dynamics, this proposal yields the Arrhenius equation for the escape rates. Then, we determine the dependence of the prefactor and the activation energy of the Arrhenius equation with re-

spect to the clustering coefficient. This information allows us to show the effect of the clustering coefficient on the robustness of a state explicitly. As the state of collective behavior is of great interested, we also study the mean first-passage time from a strongly disorder state to the state of collective behavior. Such study, in addition to the robustness of the state, may provide further understandings about the role of fluctuation in the process of reaching the state of collective behavior.

This paper is organized as follows. In Sec. II we define the *LMR*, classify its equilibrium states, and briefly describe the generating processes for the networks used in the numerical study. Based on the *LMR*, we numerically calculate the occurrence probabilities of different classes of equilibrium states for the systems starting with strongly disorder states, and the results as functions of clustering coefficient are shown in Sec. III where the dependence on the system size for the occurrence probability is also discussed. In Sec. IV, we first introduce the stochastic *LMR*, then the analysis on the escape rates of different classes of equilibrium states based on the stochastic dynamics are given. Moreover, the results for the mean first-passage time to the state of system-wide coordination are also presented in this section. Finally, a summary of the results and some general conclusions are given in Sec. V.

2 Deterministic Dynamics and Networks

We first specify the *LMR* and classify the corresponding equilibrium states. Consider a network system with the distribution of edges given by an $N \times N$ adjacency matrix

A. Here the matrix A is symmetric with the elements $a_{ij} = 1$ for the connected sites i and j, and 0 otherwise. The dynamic variable associated with a site i is denoted as x_i, which takes two possible values, either 1 or -1. The system evolves from an initial to a new configuration in discrete time step according to *LMR* whose operation can be either synchronous or asynchronous. In this work, we consider the synchronous dynamics for which, the rule can be written as

$$x_i\left(t+1\right) = \mathbf{sgn}\left(\sum_{j=1}^{N} a_{ij}x_j\left(t\right)\right) \qquad (1)$$

for $i = 1, ..., N$, where the **sgn** function is a standard threshold function with $\mathbf{sgn}(x) = +1$ for $x > 0$ and -1 for $x < 0$, and we set $x_i\left(t+1\right) = x_i\left(t\right)$ for $\sum_{j=1}^{N} a_{ij}x_j\left(t\right) = 0$. The dynamics of Eq. (1) has been widely studied in discrete neural networks, and the existence of equilibrium states can be shown by employing the Lyapunov energy function[6,15],

$$E\left(t\right) = -\sum_{i=1}^{N} x_i\left(t\right)\left[\sum_{j=1}^{N} a_{ij}x_j\left(t-1\right)\right]. \qquad (2)$$

Moreover, the period of an equilibrium state is either 1 or 2[15].

The number of equilibrium states indicates the capacity of a neural network. But, we are interested in the occurrence of the state of collective behavior and the global characters of other equilibrium states in case that the collective behavior can not be reached by systems. Thus, the equilibrium states of period-1 are divided into two classes, S_0 and S_1. Here the class S_0 is for the states of collective behavior for which, all node-states have the same value, either 1 or -1; and the class S_1 consists of all trapped

states. For the equilibrium states of period-2, the system oscillates between a pair of states, and we refer all pairs of equilibrium states as the class S_2. As the order parameter is defined as $M_s = \sum_{i=1}^{N} x_i/N$, the two states of S_0 have the value 1 and -1, respectively, and the states of S_1 and S_2 have $|M_s| < 1$.

Two types of networks, Watts-Strogatz (WS) and SF networks, are used to define the adjacency matrix A in the dynamics of Eq. (1). The WS networks are made from a regular lattice for which, N sites are placed around a circle and each site has degree, say k_0, connecting to the right and to the left symmetrically, then a probability p is assigned to rewire the edges randomly[16]. As the p value increases from 0 to 1, the resultant network changes from a regular lattice to a random graph with the clustering coefficient C decreasing from the highest value down to k_0/N . Here the C value of a network is defined as the average of the clustering coefficients associated with all sites, and the clustering coefficient of a site, say i, is given as

$$C_i = \frac{2y_i}{k_i (k_i - 1)}, \qquad (3)$$

where y_i is the number of existent edges between the k_i neighbors of the site i.

The SF networks are a special category of networks for which, the degree distributions take the form of power low as $P(k) \sim k^{-\gamma}$ with decay exponent γ. A conventional way of generating a SF network is the scheme of preferential attachment proposed by Barabasi and Albert[11], and it yields a network with $\gamma \approx 2.9$ and small C value $C \approx N^{-0.75}$[12]. The SF networks with different γ and

C values are employed for numerical study in this work, and they were generated by using the modified schemes of preferential attachment, proposed by Holme et al.[17] and by Leary et al.[18], in a systematic way. For tuning the C value without altering the γ value, a step, called triad-formation, is added to the process of preferential attachment with a assigned probability; the larger the assigned probability of performing triad-formation, the larger the C value of the resultant network is[17]. Alternatively, we alter the γ value without affecting the C value by switching the uniform distribution for the random numbers used in the process of preferential attachment to the distribution of a designed probability density function[18]. As the designed probability density function further enhances the probability of connecting a new edge to a site with large degree, we obtain a network with a larger γ value. On the other hand, for the opposite tendency in the designed function the resultant network has a smaller γ value.

3 Occurrence Probabilities of Equilibrium States

We first calculate numerically the occurrence probability for three different classes of equilibrium states of Eq. (1) to analyze the clustering effect. The occurrence probabilities denoted as P_i for the class S_i of the equilibrium states with $i = 0$, 1, or 2 for systems modelled by the WS and the SF networks. To minimize statistical errors in the simulation results, we generate 1000 samples for the

WS networks with a given value of rewiring probability, and the corresponding clustering coefficient $\overline{C}$ is defined as the average of the coefficients of all samples. Then, the P_i value associated with a $\overline{C}$ is given by the fractional percentage of occurrence for the equilibrium states belonging to the class S_i over the 10^6 trajectories. Here the trajectories all start from strongly disorder states and are equally distributed in the 1000 samples.

The results of P_i for the *WS* networks are shown as the plots of P_i vs. $\overline{C}$ in Fig. 1(a) for different k_0 values with $N = 1000$ and in Fig. 1(b) for different N values with $k_0 = 8$. Here the $\overline{C}$ values are in the range $0 < \overline{C} < 0.5$ for which, the corresponding rewiring probabilities are ranged between $0.04 < p < 0.5$; this is also the range of $\overline{C}$ generated for the *SF* networks. As shown in the plots, systems are more likely to be led to the states of collective behavior for small $\overline{C}$ values $(\overline{C} \sim 0.05)$, to the equilibrium states of S_1 for medium $\overline{C}$ values $(\overline{C} \sim 0.15)$, and to the equilibrium states of S_2 for high $\overline{C}$ values $(\overline{C} \sim 0.3)$. The results also indicate that the P_0 value is suppressed and the P_2 value is enhanced as the node number N increases, and the tendency is opposite for increasing the average number of degree k_0.

Analytic expressions $P_i\left(N, \overline{C}\right)$, which can fit the data shown in Fig. 1(b) properly, are found for a better understanding about the N-dependence of the occurrence probabilities P_i. The results are

$$P_0\left(N, \overline{C}\right) = \frac{1.11 - (0.06) N^{0.16}}{1 + \exp\left\{\left[91.53 - (407.28)/N^{0.35}\right]\overline{C} - 6.70\right\}},$$
(4)

$$P_2\left(N, \overline{C}\right) = 1.03 - \frac{0.95 + \left(1.04 \times 10^5/N^{2.07}\right) z\left(N, \overline{C}\right)}{1 + z\left(N, \overline{C}\right)},$$
(5)

and

$$P_1\left(N, \overline{C}\right) = 1 - P_0\left(N, \overline{C}\right) - P_2\left(N, \overline{C}\right),$$
(6)

where $z\left(N, \overline{C}\right)$ is

$$z\left(N, \overline{C}\right) = \left[\frac{\overline{C}}{\left(0.65/N^{0.07} - 0.25\right)}\right]^5.$$
(7)

As shown as the solid lines in Fig. 1(b), the expressions agree with the numerical data very well. Eq. (4) reveals an important feature about P_0, that is, the P_0 value is suppressed as increasing N or $\overline{C}$. Consequently, there exists a site number $N^*\left(k_0\right)$ that $P_0 \approx 0$ for $N > N^*\left(k_0\right)$, where $N^*\left(k_0\right)$ decreases as k_0 decreases. Moreover, Eq. (5) indicates that the P_2 value increases as N increases. Thus, we may have $P_0 \approx 0$, $P_1 \approx 0$, and $P_2 \approx 1$ for $N > N^*\left(k_0\right)$. This is demonstrated by the numerical results shown in Fig. 2 where the P_i values as functions of $\overline{C}$ for $N = 6000$ and $k_0 = 4$ are given.

The statistics for the numerical study in the *SF* networks is similar to the case of the *WS* networks. As the modified schemes of preferential attachment are applied, the γ value may change slightly in tuning the C value[17], and the C value may alter slightly in tuning the γ value[18]. Thus, the samples are characterized by the set-values $\left(\overline{\gamma}, \overline{C}\right)$ with 1000 members belonging to a $\left(\overline{\gamma}, \overline{C}\right)$. There are three distinct $\overline{\gamma}$ values, $\overline{\gamma} = 2.82, 2.55$, and 2.29, and the possible $\overline{C}$ values for a $\overline{\gamma}$ value are in the range between 0 and 0.5. All samples have the site number $N = 5000$ and the average degree of a site $\langle k \rangle = 4$. The P_i value is the result over the 10^6 trajectories for which, they are equally distributed

in the 1000 samples of the SF networks belonging to the same $(\overline{\gamma}, \overline{C})$ and each trajectory starts from a strongly disorder state. The numerical results are shown as the plots of P_i vs. $\overline{C}$ in Fig. 3 for three different $\overline{\gamma}$ values.

The results obtained from the SF networks indicate that although the $\overline{\gamma}$ value may affect significantly the convergent speed of the system leading to a equilibrium state, it has little effect on the P_i value for which, the $\overline{C}$ value plays a major role. Moreover, similar to the case of the WS networks, as shown in Fig. 3 the clustering coefficient $\overline{C}$ drives the system from the state of collective behavior at low $\overline{C}$ $(\overline{C} \leq 0.1)$ to the the phase of oscillation between two states at high $\overline{C}$ $(0.5 > \overline{C} > 0.3)$, and the system is trapped in a state of S_1 at medium $\overline{C}$. One may further expect that the N-dependence for P_i has the same feature qualitatively as that for the WS networks.

4 Stochastic Dynamics

Noise is a very natural component for real systems, its physical origins can be traced to incomplete information, processing errors, or other environmental perturbations. To add a component of noises to the LMR, we propose a stochastic version based on the assumption that the effect of noise is localized and appears as the fluctuation in recognizing the value of a node-state by its connected neighbors. Then, the stochastic LMR is given as

$$x_i(t+1) = \mathbf{sgn}\left(\sum_{j=1}^{N} a_{ij}\mathbf{sgn}\left(x_j(t) + \sqrt{2D}\xi_j(t)\right)\right),$$

$$(8)$$

where $\xi_i(t)$ is the Gaussian white noise with the zero mean and the δ-function correlation, i,e, $\langle\xi_i(t)\rangle = 0$ and $\langle\xi_i(t)\xi_j(s)\rangle = \delta_{i,j}\delta(t-s)$, and D is the diffusion constant which characterizes the strength of noise. The equilibrium states of Eq. (1) become transient for the stochastic dynamics of Eq. (8). Consequently, the escape time for different classes of the equilibrium states can be measured, the results are denoted as $\langle\tau^{(i)}\rangle$ with $i = 0$, 1, and 2 for the class S_0, S_1, and S_2, respectively, and the inverse of $\langle\tau^{(i)}\rangle$ yields the escape rate $\langle\kappa_i\rangle$, where the bracket of $\tau^{(i)}$ or κ_i represents the average of the results over the samples of different networks.

As the stochastic dynamics of Eq. (8) is applied to the equilibrium states of Eq. (1), one can expect that the escape rates obey the Arrhenius equation[19],

$$\langle\kappa_i\rangle = A_i \exp\left(-\frac{\Delta G_i}{D}\right), \qquad (9)$$

where A_i is the prefactor, and ΔG_i is the activation energy. Note that based on the fluctuation-dissipation theorem we may identify $D = k_B T$ with the Boltzman constant k_B and the absolute temperature T[20]. The factor A_i, which is equivalent to the rate constant in a chemical reaction, signifies the entropy effect, and one can expect that it has a strong dependence on the clustering coefficient. The value of ΔG_i gives the maximum potential energy required to escape from the equilibrium state. Thus, the results of A_i and ΔG_i provide insights on the robustness of the equilibrium states of different classes as the geometric structure of a system varies.

We perform the numerical measurements for $\langle\kappa_i\rangle$ in the SF networks, and the statistics for the measurements

is described as follows. We first single out the m_0 equilibrium states belonging to the class S_i for a set-value $\left(\overline{\gamma}, \overline{C}\right)$ with $m_0 = 25000$, where the m_0 states of the class S_0 are identical as $x_i = +1$ for all nodes. Then, the number of time-steps required to escape from each of the m_0 states is measured with a preassigned cut-off time-step $t_{\max}$, such a measurement is repeated for 20 times to realize the Gaussian distribution of the noise $\xi_i(t)$, and the escape time-step $\left\langle \tau^{(i)} \right\rangle$ for the class S_i is given by the average value over the 20 simulations and over the m_0 states.

Our results for the SF networks with $\left(\overline{\gamma}, \overline{C}\right) = (2.82, 0.3026)$ are shown in Fig. 3 as the plots of $\log \left\langle \tau^{(i)} \right\rangle$ vs. $1/D$ with $i = 0$, 1, and 2, and those for different $\left(\overline{\gamma}, \overline{C}\right)$ values also yield straight lines in the same plots. The results indicate that the escape rates $\left\langle \kappa_i \right\rangle$ for the equilibrium states belonging to the class S_i agree with Eq. (9), such agreement persists as the cut-off time-steps $t_{\max}$ increases from 10000 to 20000 as shown in Fig. 4. Then, the activation energy ΔG_i of Eq. (9) is determined from the slope of $\log \left\langle \tau^{(i)} \right\rangle$ vs. $1/D$, it yields $\Delta G_0 = 0.81$ and $\Delta G_1 = \Delta G_2 = 1.0$, independent of the $\left(\overline{\gamma}, \overline{C}\right)$ value. Moreover, the prefactor A_i can be determined by the intersection between the straight line of $\log \left\langle \tau^{(i)} \right\rangle$ vs. $1/D$ and the vertical line of $1/D = 0$, we have $A_0 = 97.08$, which is independent of the $\left(\overline{\gamma}, \overline{C}\right)$ value, and the other two prefactors A_1 and A_2 depend on the network geometry. The $\overline{\gamma}$-dependence may occur for A_{S_1} when $\overline{C}$ is large, for example, we have $A_{S_1} = 10.11$, 7.20, and 5.88 for $\overline{C} = 0.3$ but $\overline{\gamma} = 2.29$, 2.55, and 2.82, respectively; such dependence is not found for A_{S_2}. On the other hand, both A_1 and A_2 are very sensitive to the $\overline{C}$

value, and their $\overline{C}$-dependence is shown explicitly in Fig. 5 for $\overline{\gamma} = 2.82$.

Some important features for Eq. (9) are revealed from the results shown in Figs. 4 and 5. Firstly, the decay rate for the state of collective behavior is universal in the sense that it is independent of the network geometry, so are the ΔG_i values for the equilibrium states belonging to the classes S_1 and S_2. Moreover, the activation energy for the states of S_1 and S_2 is larger than that for the states of S_0. For the entropy effect on the decay rate, the $\overline{C}$ value has a significant role in determining the values of A_1 and A_2. As the results of Fig. 5 indicate, the A_2 value decreases for increasing the $\overline{C}$ value, and the A_1 value behaves oppositely; moreover, the equilibrium states of S_2 possess a very large prefactor, and this overcomes the higher activation energy and renders the states to be very fragile. Then, as the conceptual sketch shown in Fig. 6 for the potential barriers of the equilibrium states belonging to different classes, we may conclude that the states of S_2 are very easy to break up, and the states of S_1 are the most robust among the equilibrium states of three classes.

As the state of collective behavior is of great interested, we then study its mean first-passage time for the system with the SF networks in a noisy environment. In the numerical calculations, we first assign a strongly disorder configuration to the system, then follow the stochastic LMR of Eq. (8) to generate a trajectory and to record the time-steps for the first appearance of a state of S_0. Here, the recorded time-steps is set as $t_{\max} = 10^4$ for the absence of the states of S_0 at $t = t_{\max}$. We generate 10^6

trajectories totally for a given set of $(\overline{\gamma}, \overline{C})$, and the mean first-passage time, denoted as $\langle \tau_0 \rangle$, is given as the average of the recorded time-steps over all trajectories. As the differences in $\langle \tau_0 \rangle$ caused by different γ values are insignificant, we show the results as the plot of $\langle \tau_0 \rangle$ vs. D for $\overline{\gamma} = 2.82$ and $\overline{C} = 0.0066, 0.2016$, and 0.3026, respectively in Fig. 7. It is interesting to observe that the $\langle \tau_0 \rangle$ value as a function of D is non-monotonic. As the D value increases, the $\langle \tau_0 \rangle$ value first decreases, reaches a minimum flat valley in the range of $0.07 \lesssim D \lesssim 0.22$ with $\langle \tau_0 \rangle < 10$, then increases abruptly at $D_{\max} \simeq 0.22$, and the state of collective behavior becomes unreachable for $D > D_{\max}$. Moreover, the role of clustering coefficient in the mean first-passage time is not very significant, as the results shown in Fig. 5 indicate, the difference in clustering coefficients is noticed by the mean first-passage time for $D \leq 0.10$.

5 Summary and Conclusion

In summary, we show explicitly how the clustering coefficient of a system affect the occurrence probability of the different types of equilibrium states associated with the LMR. As the state of system-wide coordination is concerned, the increasing of the clustering coefficient would suppress its probability of occurrence. On the other hand, systems with large clustering coefficients are easily led to trapped states or oscillations between pairs of states. We also propose a stochastic version of the LMR for which, the decay rate of an equilibrium state obeys the Arrhenius equation. This allows us to quantify the robustness of the equilibrium states through the values of the prefactor and the activation energy in the Arrhenius equation. Our results indicate that the states of period-2 are very fragile, and the trapped states are the most robust among the three types. For systems in noisy environments, our results obtained from the stochastic LMR indicate that there exists a range of noise for which, the mean first-passage time from strongly disorder states to the states of system-wide coordination is the shortest. Thus, the efficiency in reaching the state of collective behavior may be improved for systems with certain amount of noise. As the distribution of subgroups in a social system can be characterized by the clustering coefficient, all these results may provide wide applications in the study of collective behaviors of social systems.

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